ALEXANDROV EMBEDDED CLOSED MAGNETIC GEODESICS ON S^2

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Abstract. We prove the existence of Alexandrov embedded closed magnetic geodesics on any two dimensional sphere with nonnegative Gauß curvature.

1. Introduction

Let (S^2, g) be the two dimensional sphere equipped with a smooth metric g and $k : S^2 \to \mathbb{R}$ a smooth positive function. We consider the following two equations for curves γ on S^2:

$$D_{t,g}\dot\gamma = k(\gamma)J_g(\gamma)\dot\gamma, \tag{1.1}$$

and

$$D_{t,g}\dot\gamma = |\dot\gamma|_g k(\gamma)J_g(\gamma)\dot\gamma, \tag{1.2}$$

where $D_{t,g}$ is the covariant derivative with respect to g, and $J_g(x)$ is the rotation by $\pi/2$ in T_xS^2 measured with g.

Equation (1.1) describes the motion of a charge in a magnetic field corresponding to the magnetic form kdV_g and solutions to (1.1) will be called *(k-)magnetic geodesics* (see [1,3,5]). Equation (1.2) corresponds to the problem of *prescribing geodesic curvature*, as its solutions γ are constant speed curves with geodesic curvature $k_g(\gamma, t)$ given by $k(\gamma(t))$.

It is easy to see that a nonconstant magnetic geodesic γ lies in a fixed energy level E_c, i.e. there is $c > 0$, such that

$$(\gamma, \dot\gamma) \in E_c := \{(x, V) \in TS^2 : |V|_g = c\}.$$

For fixed k and $c > 0$ the equations (1.1) and (1.2) are equivalent in the following sense: If γ is a nonconstant solution of (1.2) with k replaced by k/c, then the curve $\gamma_c(t) := \gamma(ct/|\dot\gamma|_g)$ is a k-magnetic geodesic in E_c, and a k-magnetic geodesic in E_c solves (1.2) with k replaced by k/c.

We study the existence of closed curves with prescribed geodesic curvature or equivalently the existence of periodic magnetic geodesics on prescribed energy levels E_c.

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Solutions to (1.1) or (1.2) are invariant under a circle action: For $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ and a closed curve γ we define a closed curve $\theta * \gamma$ by

$$\theta * \gamma(t) = \gamma(t+\theta).$$

Thus, any solution gives rise to a S^1-orbit of solutions and we say that two solutions γ_1 and γ_2 are (geometrically) distinct, if $S^1 * \gamma_1 \neq S^1 * \gamma_2$.
There are different approaches to this problem, the Morse-Novikov theory for (possibly multi-valued) variational functionals (see [7,13,14]), the theory of dynamical systems using methods from symplectic geometry (see [1,4–6,10]) and Aubry-Mather's theory (see [3]), and recently the theory of vector fields on infinite dimensional manifolds (see [11]).
We follow [11] and consider solutions to (1.2) as zeros of the vector field $X_{k,g}$ defined on the Sobolev space $H^{2,2}(S^1, S^2)$ as follows: For $\gamma \in H^{2,2}(S^1, S^2)$ we let $X_{k,g}(\gamma)$ be the unique weak solution of

$$\big(-D_{t,g}^2 + 1\big)X_{k,g}(\gamma) = -D_{t,g}\dot\gamma + |\dot\gamma|_g k(\gamma) J_g(\gamma)\dot\gamma \tag{1.3}$$

in $T_\gamma H^{2,2}(S^1, S^2)$. The uniqueness implies that any zero of $X_{k,g}$ is a weak solution of (1.2) which is a classical solution in $C^2(S^1, S^2)$ applying standard regularity theory.
Concerning (S^2, g) and a positive function k it is conjectured,

$$\text{every positive energy level } E_c \text{ contains a } k\text{-magnetic geodesic.} \tag{1.4}$$

More precisely, the open problem in [2, 1994-35,1996-18] is to show the existence of at least two closed k-magnetic geodesics on every positive energy level, which is is true for small energy levels depending on g and k (see [5,6]).
In [11] it is shown that (1.4) is true, if the metric g is $\frac{1}{4}$-pinched, i.e. the Gauß curvature K_g satisfies

$$\sup K_g < 4 \inf K_g.$$

In fact, if g is $\frac{1}{4}$-pinched and k is a positive function, then every positive energy level E_c contains an embedded (simple) closed k-magnetic geodesic and the number of embedded closed k-magnetic geodesics in E_c is even, provided they are all nondegenerate. We shall extend the above existence results. Instead of working in the class of embedded curves we consider solutions, that are Alexandrov embedded.

Definition 1.1. *(oriented Alexandrov embedded) Let $B \subset \mathbb{R}^2$ denote the open ball of radius* 1 *centered at* $0 \in \mathbb{R}^2$*. An immersion $\gamma \in C^1(\partial B, S^2)$ will be called* oriented Alexandrov embedded*, if there is an immersion $F \in C^1(\overline{B}, S^2)$, such that $F|_{\partial B} = \gamma$ and F is orientation preserving in the sense that*

$$\langle DF|_x x, J_g(\gamma(x))\dot\gamma(x)\rangle_{T_{\gamma(x)}S^2, g} > 0$$

for all $x \in \partial B$.

Usually, Alexandrov embedded curves are defined to be the boundary of immersed manifolds. We restrict ourselves to the case of the ball B and oriented immersions F. If we equip B with the metric F^*g induced by F, then the outer normal $N_B(x)$ at $x \in \partial B$ with respect to F^*g satisfies

$$DF|_x N_B(x) = N_\gamma(x)$$

where $N_\gamma(x)$ denotes the normal to the curve γ at $x \in \partial B$ defined by

$$N_\gamma(x) := |\dot\gamma(x)|^{-1} J_g(\gamma(x))\dot\gamma(x).$$

We shall prove

Theorem 1.2. *Let g be a smooth metric on S^2 with nonnegative Gauß curvature and k a positive smooth function. Then there is an oriented Alexandrov embedded curve $\gamma \in C^2(S^1, S^2)$ that solves* (1.2) *and the number of oriented Alexandrov embedded solutions to (1.2) is even provided they are all nondegenerate.*

The equivalence between (1.1) and (1.2) leads to

Corollary 1.3. *Let g be a smooth metric on S^2 with nonnegative Gauß curvature and k a positive smooth function. Then every energy level E_c contains an oriented Alexandrov embedded magnetic geodesic and the number of oriented Alexandrov embedded magnetic geodesics in E_c is even provided they are all nondegenerate.*

The proof of our existence results is organized as follows. After setting up notation in Section 2 and introducing the classes of maps and spaces needed for our analysis we recall in Section 3 the definition and properties of the S^1-equivariant Poincaré-Hopf index,

$$\chi_{S^1}(X_{k,g}, M_A) \in \mathbb{Z},$$

where M_A is the set of oriented Alexandrov embedded regular curves in $H^{2,2}(S^1, S^2)$. For positive constants k_0 we shall show that

$$\chi_{S^1}(X_{k_0,g_{can}}, M_A) = -2,$$

where g_{can} denotes the round metric induced by $S^2 = \partial B_1(0) \subset \mathbb{R}^3$. Section 5 contains the apriori estimate which implies that the set of solutions to (1.2) is compact in M_A, if the Gauß curvature of (S^2, g) is nonnegative. The homotopy invariance of the S^1-equivariant Poincaré-Hopf index then leads to the identity

$$\chi_{S^1}(X_{k,g}, M_A) = \chi_{S^1}(X_{k_0,g_{can}}, M_A) = -2.$$

The resulting proof of Theorem 1.2 is given in Section 6.

2. Preliminaries

Let $S^2 = \partial B_1(0) \subset \mathbb{R}^3$ be the standard round sphere with induced metric g_{can} and orientation such that the rotation $J_{g_{can}}(y)$ is given for $y \in S^2$ by

$$J_{g_{can}}(y)(v) := y \times v \text{ for all } v \in T_y S^2,$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. If we equip S^2 with a general Riemannian metric g, then the rotation by $\pi/2$ measured with g is given by

$$J_g(y)v = \big(G(y)\big)^{-1} J_{g_{can}}(y)\big(G(y)\big)v \quad \forall v \in T_y S^2,$$

where $G(y)$ denotes the positive symmetric map $G(y) \in \mathcal{L}(T_y S^2)$ satisfying

$$\langle v, w \rangle_{T_y S^2, g} = \langle G(y)v, G(y)w \rangle_{T_y S^2, g_{can}} \quad \forall v, w \in T_y S^2.$$

The geodesic curvature $k_g(\gamma, t)$ of an immersed curve γ at t is defined by

$$k_g(\gamma, t) := |\dot\gamma(t)|_g^{-2} \big\langle \big(D_{t,g}\dot\gamma\big)(t), N_g(\gamma(t)) \big\rangle_g,$$

where $N_g(\gamma(t))$ denotes the unit normal of γ at t given by

$$N_g(\gamma(t)) := |\dot\gamma(t)|_g^{-1} J_g(\gamma(t))\dot\gamma(t).$$

The vector field $X_{k,g}$ as well as the set of solutions to (1.2) is invariant under a circle action: For $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ and $\gamma \in H^{2,2}(S^1, S^2)$ we define $\theta * \gamma \in H^{2,2}(S^1, S^2)$ by

$$\theta * \gamma(t) = \gamma(t + \theta).$$

Moreover, for $V \in T_\gamma H^{2,2}(S^1, S^2)$ we let

$$\theta * V := V(\cdot + \theta) \in T_{\theta * \gamma} H^{2,2}(S^1, S^2).$$

Then $X_{k,g}(\theta * \gamma) = \theta * X_{k,g}(\gamma)$ for any $\gamma \in H^{2,2}(S^1, S^2)$ and $\theta \in S^1$. Thus, any zero gives rise to a S^1-orbit of zeros.
We consider for $m \in \mathbb{N}_0$ the set of Sobolev functions

$$H^{m,2}(S^1, S^2) := \{\gamma \in H^{m,2}(S^1, \mathbb{R}^3) : \gamma(t) \in \partial B_1(0) \text{ for a.e. } t \in S^1.\}$$

For $m \geq 1$ the set $H^{m,2}(S^1, S^2)$ is a sub-manifold of the Hilbert space $H^{m,2}(S^1, \mathbb{R}^3)$ and is contained in $C^{m-1}(S^1, \mathbb{R}^3)$. Hence, if $m \geq 1$ then $\gamma \in H^{m,2}(S^1, S^2)$ satisfies $\gamma(t) \in \partial B_1(0)$ for all $t \in S^1$. In this case the tangent space $T_\gamma H^{m,2}(S^1, S^2)$ of $H^{m,2}(S^1, S^2)$ at $\gamma \in H^{m,2}(S^1, S^2)$ is given by

$$T_\gamma H^{m,2}(S^1, S^2) := \{V \in H^{m,2}(S^1, \mathbb{R}^3 : V(t) \in T_{\gamma(t)} S^2 \text{ for all } t \in S^1\}.$$

For $m = 0$ the set $H^{0,2}(S^1, S^2) = L^2(S^1, S^2)$ fails to be a manifold. In this case we define for $\gamma \in H^{1,2}(S^1, S^2)$ the space $T_\gamma L^2(S^1, S^2)$ by

$$T_\gamma L^2(S^1, S^2) := \{V \in L^2(S^1, \mathbb{R}^3 : V(t) \in T_{\gamma(t)} S^2 \text{ for a.e. } t \in S^1\}.$$

A metric g on S^2 induces a metric on $H^{m,2}(S^1, S^2)$ for $m \geq 1$ by setting for $\gamma \in H^{m,2}(S^1, S^2)$ and V, $W \in T_\gamma H^{m,2}(S^1, S^2)$

$$\langle W, V\rangle_{T_\gamma H^{m,2}(S^1,S^2),g} := \int_{S^1} \Big\langle \big((-1)^{\llcorner \frac{m}{2} \lrcorner}(D_{t,g})^m + 1\big)V(t), \big((-1)^{\llcorner \frac{m}{2} \lrcorner}(D_{t,g})^m + 1\big)W(t)\Big\rangle_{\gamma(t),g} dt,$$

where $\llcorner m/2 \lrcorner$ denotes the largest integer that does not exceed $m/2$.
Let X be a differentiable vector field on $H^{2,2}(S^1, S^2)$. Then the covariant (Frechet) derivative $D_g X$,

$$D_g X : TH^{2,2}(S^1, S^2)) \to TH^{2,2}(S^1, S^2),$$

of the vector field X with respect to the metric induced by g is defined as follows: For $\gamma \in H^{2,2}(S^1, S^2)$ and $V \in T_\gamma H^{2,2}(S^1, S^2)$ we consider a C^1-curve

$$(-\varepsilon, \varepsilon) \ni s \mapsto \gamma_s \in H^{2,2}(S^1, S^2)$$

satisfying

$$\gamma_0 = \gamma \text{ and } \frac{d}{ds}\gamma_s|_{s=0} = V,$$

and define

$$D_g X|_\gamma[V](t) := D_{g,s}\Big(X\big(\gamma_s(t)\big)\Big)|_{s=0}.$$

For the vector field theory on infinite dimensional manifolds it is convenient to work with Rothe maps instead of compact perturbations of the identity, because the class of Rothe maps is open in the space of linear continuous maps. We recall the definition and properties of Rothe maps given in [16] for the sake of the readers convenience. For a Banach space E we denote by $\mathcal{GL}(E)$ the set of invertible maps in $\mathcal{L}(E)$ and by $\mathcal{S}(E)$ the set

$$\mathcal{S}(E) = \{T \in \mathcal{GL}(E) : (tT + (1-t)I) \in \mathcal{GL}(E) \text{ for all } t \in [0,1]\}.$$

Then the set of Rothe maps $\mathcal{R}(E)$ is defined by

$$\mathcal{R}(E) := \{A \in \mathcal{L}(E) : A = T + C, T \in \mathcal{S}(E) \text{ and } C \text{ compact}\}.$$

The set $\mathcal{R}(E)$ is open in $\mathcal{L}(E)$ and consists of Fredholm operators of index 0. Moreover, $\mathcal{GR}(E) := \mathcal{R}(E) \cap \mathcal{GL}(E)$ has two components, $\mathcal{GR}^\pm(E)$, with $I \in \mathcal{GR}^+(E)$. For $A \in \mathcal{GR}(E)$ we let

$$\mathrm{sgn} A = \begin{cases} +1 & \text{if } A \in \mathcal{GR}^+(E), \\ -1 & \text{if } A \in \mathcal{GR}^-(E). \end{cases}$$

If $A = I + C \in \mathcal{GL}(E)$, where C is compact, then $A \in \mathcal{GR}(E)$ and $\mathrm{sgn}A$ is given by the the usual Leray-Schauder degree of A.
Since g and k are smooth, $X_{k,g}$ is a smooth vector field (see [11, 15, Sec. 6]) on the set $H^{2,2}_{reg}(S^1, S^2)$ of regular curves,

$$H^{2,2}_{reg}(S^1, S^2) := \{\gamma \in H^{2,2}(S^1, S^2) : \dot\gamma(t) \neq 0 \text{ for all } t \in S^1\}.$$

Furthermore, we call γ a prime curve, if the isotropy group

$$\{\theta \in S^1 : \theta * \gamma = \gamma\}$$

of γ is trivial. From [11] there holds

$$\begin{aligned}
&\big(-D_{t,g}^2 + 1\big) D_g X_{k,g}|_\gamma(V) \\
&\quad = -D_{t,g}^2 V - R_g\big(V, \dot\gamma\big)\dot\gamma + |\dot\gamma|_g^{-1} \langle D_{t,g} V, \dot\gamma\rangle_g k(\gamma) J_g(\gamma)\dot\gamma \\
&\qquad + |\dot\gamma|_g \big(k'(\gamma)V\big) J_g(\gamma)\dot\gamma + |\dot\gamma|_g k(\gamma)\Big(\big(D_g J_g|_\gamma V\big)\dot\gamma + J_g(\gamma) D_{t,g} V\Big).
\end{aligned} \tag{2.1}$$

We note that (see also [16, Thm. 6.1])

$$\big(- D_{t,g}^2 + 1\big) D_g X_{k,g}|_\gamma(V) = (-D_{t,g}^2 + 1)V + T(V),$$

where T is a linear map from $T_\gamma H^{2,2}(S^1, S^2)$ to $T_\gamma L^2(S^1, S^2)$ that depends only on the first derivatives of V and is therefore compact. Taking the inverse $(-D_{t,g}^2 + 1)^{-1}$ we deduce that $D_g X_{k,g}|_\gamma$ is the form *identity* + *compact* and thus a Rothe map.
For $m \geq 1$ the exponential map $Exp_g : TH^{m,2}(S^1, S^2) \to H^{m,2}(S^1, S^2)$ is defined for $\gamma \in H^{m,2}(S^1, S^2)$ and $V \in T_\gamma H^{m,2}(S^1, S^2)$ by

$$Exp_{\gamma,g}(V)(t) := Exp_{\gamma(t),g}(V(t)),$$

where $Exp_{z,g}$ denotes the exponential map on (S^2, g) at $z \in S^2$. Due to its pointwise definition

$$\theta * Exp_{\gamma,g}(V)(t) = Exp_{\theta*\gamma,g}(\theta * V)(t).$$

3. The S^1-Poincaré-Hopf index

We define for $\gamma \in H^{2,2}(S^1, S^2)$ the form $\omega_g(\gamma) \in (T_\gamma H^{2,2}(S^1, S^2))^*$ by

$$\omega_g(\gamma)(V) := \int_0^1 \langle \dot\gamma(t), \big(- (D_{t,g})^2 + 1\big) V(t)\rangle_g \, dt = \langle \dot\gamma, V\rangle_{T_\gamma H^{1,2}(S^1,S^2),g}.$$

From Riesz' representation theorem there is $W_g(\gamma) \in T_\gamma H^{2,2}(S^1, S^2)$ such that

$$\omega_g(\gamma)(V) = \langle V, W_g(\gamma)\rangle_{T_\gamma H^{2,2}(S^1,S^2),g} \; \forall V \in T_\gamma H^{2,2}(S^1, S^2),$$

and

$$\langle W_g(\gamma)\rangle^\perp = \langle\dot\gamma\rangle^{\perp, H^{1,2}} \cap T_\gamma H^{2,2}(S^1, S^2). \tag{3.1}$$

Hence

$$W_g(\gamma) = (-(D_{t,g})^2 + 1)^{-1}\dot\gamma$$

and W_g is a C^2 vector field on $H^{2,2}(S^1, S^2)$. The form $\omega_g(\gamma)$ and the vector $W_g(\gamma)$ are equivariant under the S^1-action in the sense that for all $\theta \in S^1$ and $V \in T_\gamma H^{2,2}(S^1, S^2)$ we have

$$w_{\theta*\gamma,g}(\theta * V) = \omega_g(\gamma)(V) \text{ and } W_{\theta*\gamma,g} = \theta * W_g(\gamma).$$

We will compute the Poincaré-Hopf index for the following class of vector fields.

Definition 3.1. *Let M be an open S^1-invariant subset of prime curves in $H^{2,2}(S^1,S^2)$. A C^2 vector field X on M is called (M,g,S^1)-admissible, if*

(1) *X is S^1-equivariant, i.e. $X(\theta*\gamma)=\theta*X(\gamma)$ for all $(\theta,\gamma)\in S^1\times M$.*
(2) *X is proper in M, i.e. the set $\{\gamma\in M:\ X(\gamma)=0\}$ is compact,*
(3) *X is orthogonal to W_g, i.e. $w_g(\gamma)(X(\gamma))=0$ for all $\gamma\in M$.*
(4) *X is a Rothe field, i.e. if $X(S^1*\gamma)=0$ then*

$$D_gX|_\gamma\in\mathcal{R}(T_\gamma H^{2,2}(S^1,S^2))\ \textit{and}\ Proj_{\langle W_g(\gamma)\rangle^\perp}\circ D_gX|_\gamma\in\mathcal{R}(\langle W_g(\gamma)\rangle^\perp),$$

(5) *X is elliptic, i.e. there is $\varepsilon>0$ such that for all finite sets of charts*

$$\{(Exp_{\gamma_i,g},B_{2\delta_i}(0)):\ \gamma_i\in H^{4,2}(S^1,S^2)\ \textit{for}\ 1\le i\le n\},$$

and finite sets

$$\{W_i\in T_{\gamma_i}H^{4,2}(S^1,S^2):\ \|W_i\|_{T_{\gamma_i}H^{4,2}(S^1,S^2)}<\varepsilon\ \textit{for}\ 1\le i\le n\},$$

there holds: If $\alpha\in\bigcap_{i=1}^{n}Exp_{\gamma_i,g}(B_{\delta_i}(0))\subset H^{2,2}(S^1,S^2)$ satisfies

$$X(\alpha)=\sum_{i=1}^{n}Proj_{\langle W_g(\alpha)\rangle^\perp}\circ DExp_{\gamma_i,g}|_{Exp^{-1}_{\gamma_i,g}(\alpha)}(W_i)$$

then α is in $H^{4,2}(S^1,S^2)$.

Property (4) does not depend on the particular element γ of the critical orbit $S^1*\gamma$, because from $\theta*X(\gamma)=X(\theta*\gamma)$ we get

$$D_gX|_\gamma=(-\theta*)\circ D_gX|_{\theta*\gamma}\circ(\theta*). \tag{3.2}$$

and Rothe maps are invariant under conjugacy. Concerning the regularity property (5), taking $W_i=0$, we deduce that if $X(\gamma)=0$ then $\gamma\in H^{4,2}(S^1,S^2)$. Furthermore, if $\gamma\in H^{4,2}(S^1,S^2)$ then the map $\theta\mapsto\theta*\gamma$ is C^2 from S^1 to $H^{2,2}(S^1,S^2)$. Hence, if $X(\gamma)=0$ then

$$0=D_\theta(X(\theta*\gamma))|_{\theta=0}=D_gX|_\gamma(\dot\gamma), \tag{3.3}$$

such that the kernel of $D_gX|_\gamma$ at a critical orbit $S^1*\gamma$ is nontrivial. The parameter $\varepsilon>0$ ensures that (5) remains stable under small perturbations used in the Sard-Smale lemma below. If X is a vector field orthogonal to W_g and $X(\gamma)=0$, then

$$0=D\big(\langle X(\alpha),W_g(\alpha)\rangle_{T_\alpha H^{2,2}(S^1,S^2),g}\big)|_\gamma=\langle D_gX|_\gamma,W_g(\gamma)\rangle_{T_\gamma H^{2,2}(S^1,S^2),g}$$

where the various curvature terms and terms containing derivatives of W_g vanish as $X(\gamma)=0$. Thus, $X(\gamma)=0$ implies

$$D_gX|_\gamma:\ T_\gamma H^{2,2}(S^1,S^2)\to\langle W_g(\gamma)\rangle^\perp, \tag{3.4}$$

and the projection $\mathrm{Proj}_{\langle W_g(\gamma)\rangle^\perp}$ in (4) is unnecessary.

Definition 3.2. *Let M be an open S^1-invariant subset of prime curves in $H^{2,2}(S^1,S^2)$, $S^1*\gamma\subset M$, and X a (M,g,S^1)-admissible vector field on M. The orbit $S^1*\gamma$ is called a* critical orbit *of X, if $X(\gamma)=0$.*

The orbit $S^1 * \gamma$ *is called a* nondegenerate critical orbit *of* X*, if* $X(\gamma) = 0$ *and*

$$D_g X|_\gamma : \langle W_g(\gamma)\rangle^\perp \to \langle W_g(\gamma)\rangle^\perp \tag{3.5}$$

is an isomorphism.

The nondegeneracy of a critical orbit does not depend on the choice of γ in $S^1 * \gamma$.

Definition 3.3. *Let* g_t *for* $t \in [0,1]$ *be a family of smooth metrics on* S^2*, which induces a corresponding family of metrics on* $H^{2,2}(S^1, S^2)$*, still denoted by* g_t*. Let* M *be an open* S^1*-invariant subset of prime curves in* $H^{2,2}(S^1, S^2)$ *and* X_0*,* X_1 *two vector-fields on* M *such that* X_i *is* (M, g_i, S^1)*-admissible for* $i = 0, 1$*. A* C^2 *family of vector-fields* $X(t, \cdot)$ *on* M *for* $t \in [0,1]$ *is called a* (M, g_t, S^1)*-homotopy between* X_0 *and* X_1*, if*

- $X(0, \cdot) = X_0$ *and* $X(1, \cdot) = X_1$*,*
- $\{(t, \gamma) \in [0,1] \times M : X(t, \gamma) = 0\}$ *is compact,*
- $X_t := X(t, \cdot)$ *is* (M, g_t, S^1)*-admissible for all* $t \in [0,1]$*.*

We write (M, g, S^1)*-homotopy, if the family of metrics* g_t *is constant.*

In [11] a S^1 equivariant version of the Sard-Smale lemma [9, 12] is given:

Lemma 3.4. *Let* M *be an open* S^1*-invariant subset of prime curves in* $H^{2,2}(S^1, S^2)$ *and* X *a* (M, g, S^1)*-admissible vector field on* M*. Let* $\mathcal{U}$ *be an open neighborhood of the zeros of* X*. Then there exists a* (M, g, S^1)*-admissible vector field* Y *such that* Y *has only finitely many isolated, nondegenerate zeros,* Y *equals* X *outside* $\mathcal{U}$ *and there is a* (M, g, S^1)*-homotopy connecting* X *and* Y*.*

We let M be an open S^1-invariant subset of prime curves in $H^{2,2}(S^1, S^2)$ and X a (M, g, S^1)-admissible vector field on M. We shall define the S^1-equivariant Poincaré-Hopf index $\chi_{S^1}(X, M)$ of the vector-field X with respect to the set M. We begin with the definition of the local degree of an isolated, nondegenerate critical orbit of X.
We fix a nondegenerate critical orbit $S^1 * \gamma_0$ of X in M. As X is (M, g, S^1)-admissible, $DX|_{\gamma_0} \in \mathcal{GR}(\langle W_g(\gamma_0)\rangle^\perp)$ and we define the local degree of X at $S^1 * \gamma_0$ by

$$\deg_{loc,S^1}(X, S^1 * \gamma_0) := \operatorname{sgn} D_g X|_{\gamma_0}.$$

From (3.2) the local degree does not depend on the choice of γ_0 in $S^1 * \gamma_0$.

Definition 3.5 (S^1-degree)**.** *Let* M *be an open* S^1*-invariant subset of prime curves in* $H^{2,2}(S^1, S^2)$ *and* X *be* (M, g, S^1)*-admissible. From Lemma 3.4 there is a vector field* Y*, which is* (M, g, S^1)*-homotopic to* X*, with only finitely many zeros, that are all nondegenerate. The* S^1*-equivariant Poincaré-Hopf index (or* S^1*-degree) of* X *in* M *is defined by*

$$\chi_{S^1}(X, M) := \sum_{\{S^1 * \gamma \subset M :\, Y(S^1 * \gamma) = 0\}} \deg_{loc,S^1}(Y, S^1 * \gamma).$$

In [11] it is shown that the definition does not depend on the particular choice of Y, and that the S^1-degree does not change under homotopies in the class of (M, g, S^1)-admissible vector-fields.
The unperturbed problem with $g = g_{can}$ and $k \equiv k_0$ is analyzed in [11]. There holds

$$\chi_{S^1}(X_{k_0,g_{can}}, M_0) = -2,$$

where M_0 denotes the set of embedded curves in $H^{2,2}_{reg}(S^1, S^2)$ and the zeros of $X_{k_0,g_{can}}$ are given by n-fold iterates of a S^2-family of simple curves corresponding to parallels with a radius depending on k_0. Since n-fold iterates are oriented Alexandrov embedded, if and only if $n = 1$ by Lemma 4.1 below, the zeros of $X_{k_0,g_{can}}$ in the set of oriented Alexandrov embedded regular curves M_A and M_0 coincide and thus

$$\chi_{S^1}(X_{k_0,g_{can}}, M_A) = -2. \tag{3.6}$$

4. Alexandrov embedded curves

Lemma 4.1. *For $n \in \mathbb{N}$ let γ_n be the n-fold iterate of a simple curve $\gamma_1 \in C^1(S^1, S^2)$ in (S^2, g_{can}) with nonnegative geodesic curvature. Then γ_n is oriented Alexandrov embedded if and only if $n = 1$.*

Proof. Since γ_1 is simple, $S^2 \setminus \gamma_1(S^1)$ consists of two simply connected components by the Jordan curve theorem. We let $\tilde{B}$ be the component, where the normal N_γ to γ is the outer normal. By the Riemann mapping theorem there is an orientation preserving diffeomorphism from $\tilde{B}$ to the open ball B showing that γ_1 is oriented Alexandrov embedded.
Assume γ_n is oriented Alexandrov embedded and let F_n be the corresponding immersion. We equip B with the metric $F_n^* g_{can}$ induced by F_n. To obtain a contradiction we assume that F_n is surjective. From the Gauß-Bonnet formula theorem we derive

$$\begin{aligned}
2\pi &= \int_{\partial B} k_{F_n * g_{can}}\, dS_{F_n * g_{can}} + \int_B K_{F_n^* g_{can}}\, dA_{F_n^* g_{can}} \\
&= \int_{\gamma_n} k_{g_{can}}\, ds_{g_{can}} + \int_B K_{F_n^* g_{can}}\, dA_{F_n^* g_{can}} \\
&\geq \int_{F_n(B)} K_{g_{can}}\, dA_{g_{can}} \geq \int_{S^2} K_{g_{can}}\, dA_{g_{can}} \\
&= 4\pi,
\end{aligned}$$

which leads to the desired contradiction. Hence F_n is not surjective and using stereographic coordinates we may assume that γ_1 is a simple curve in the plane $(\mathbb{R}^2, \delta)$ with standard metric δ. If we apply the Gauß-Bonnet

formula to $(B, F_n^*\delta)$ and the embedded curve γ_1 in the plane, we obtain

$$\begin{aligned}2\pi &= \int_{\partial B} k_{F_n^*\delta}\, dS_{F_n^*\delta} + \int_B K_{F_n^*\delta}\, dA_{F_n^*\delta}\\ &= \int_{\gamma_n} k_\delta\, dS_\delta = n\int_{\gamma_1} k_\delta\, dS_\delta = n2\pi,\end{aligned}$$

which is only possible for $n = 1$. $\square$

Lemma 4.2. *Let (γ_n) in $C^2(\partial B, S^2)$ be a sequence of immersions, which are oriented Alexandrov embedded, such that (γ_n) converges to an immersion γ_0 in $C^2(\partial B, S^2)$ with strictly positive geodesic curvature. Then γ_0 is oriented Alexandrov embedded.*

Proof. We fix (γ_n) and γ_0 that satisfy the above assumptions and denote by $F_n : \overline{B} \to S^2$ the corresponding sequence of oriented immersions, such that $F_n|_{\partial B} = \gamma_n$. As γ_n is a C^2-map, we may assume F_n is in $C^2(\overline{B}, S^2)$ as well. Since the convergence is in $C^2(\partial B, S^2)$, there is $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}_0$ the map $T_{S^2,n} : \partial B \times (-\varepsilon_0, \varepsilon_0) \to S^2$ defined by

$$T_{S^2,n}(y,t) := Exp_{\gamma_n(y),g}\big(tN_{S^2,n}(y)\big)$$

is an immersion, where $N_{S^2,n}(y)$ is the outer normal at $\gamma_n(y)$ to $F_n(\partial B)$, if $n \neq 0$, and $N_{S^2,0}(y) = \lim_{n\to\infty} N_{S^2,n}(y) = J_g(\gamma_0(y))|\dot\gamma_0(y)|^{-1}\dot\gamma_0(y)$. Moreover, we may assume that the geodesic curvature of the curves $T_{S^2,n}(\cdot, s)$ is uniformly bounded, i.e. there is $k_0 > 0$ such that

$$k_0 \le k_g\big(T_{S^2,n}(t,s), t\big) \le k_0^{-1}\ \forall (n,t,s) \in \mathbb{N} \times \partial B \times (-\varepsilon_0, \varepsilon_0). \tag{4.1}$$

We consider $(\overline{B}, F_n^* g)$, where $F_n^* g$ denotes the metric induced by F_n. Shrinking $\varepsilon_0 > 0$ we have the following

Proposition 4.3. *For all $n \in \mathbb{N}$ the map $T_{B,n} : \partial B \times (-\varepsilon_0, 0] \to \overline{B}$,*

$$T_{B,n}(y,t) := Exp_{y,F_n^*g}(tN_{B,n}(y)),$$

where $N_{B,n}(y)$ denotes the outer normal to ∂B at y with respect to $F_n^* g$, is well defined and a diffeomorphism onto its range.

We postpone the proof of the proposition and proceed with the proof of Lemma 4.2.
Due to the unique solvability of the ordinary differential equation corresponding to the exponential function we have

$$F_n \circ T_{B,n}(y,t) = T_{S^2,n}(y,t) \text{ for all } (y,t) \in \partial B \times (-\varepsilon_0, 0]. \tag{4.2}$$

Since (γ_n) converges to γ_0 in $C^2(\partial B, S^2)$, there is $n_0 \in \mathbb{N}$ such that a reparametrization of γ_0 is a graph over γ_{n_0}, i.e. there is $s \in C^1(\partial B, (-\varepsilon_0, \varepsilon_0))$ and an oriented diffeomorphism $\alpha \in C^2(\partial B, \partial B)$, which is close to the identity, such that

$$\gamma_0(\alpha(y)) = T_{S^2,n_0}(y, s(y)) \quad \forall y \in \partial B.$$

If we define $F_0 : \overline{B} \to S^2$ by

$$F_0(x) = \begin{cases} F_{n_0}(x), & \text{if } x \text{ is in} \\ & B \setminus T_{B,n_0}(\partial B \times (-\varepsilon_0, 0)), \\ T_{S^2,n_0}\big(y, (s(y)\varepsilon_0^{-1} + 1)t + s(y)\big), & \text{if } x = T_{B,n_0}(y,t) \text{ for} \\ & (y,t) \in (\partial B \times (-\varepsilon_0, 0]), \end{cases}$$

then $F_0 \in C^1(\overline{B}, S^2)$, by (4.2). Since $F_0|_{\partial B} = \gamma_0 \circ \alpha$ and F_0 is an oriented immersion as a composition of such immersions, we see that $\gamma \circ \alpha$ is oriented Alexandrov embedded. The diffeomorphism α can be extended to an oriented diffeomorphism A of $\overline{B}$: We consider ∂B as $\mathbb{R}/2\pi\mathbb{Z}$, assume after a rotation $\alpha(0) = 0$, and define A in polar coordinates by

$$A(r, \varphi) := \big(r, \int_0^\varphi s(r) + (1 - s(r))\alpha'(\tau)\, d\tau\big),$$

where $s \in C^\infty([0,1],[0,1])$ is any function satisfying

$$s(r) = 1, \text{ if } 0 \le r \le \frac{1}{2} \text{ and } s(r) = 0, \text{ if } r = 1$$

Consequently, γ_0 is oriented Alexandrov embedded as well using $F_0 \circ A^{-1}$, which yields the claim of Lemma 4.2.

It remains to prove Proposition 4.3: Firstly, we note that for any $n \in \mathbb{N}$ the map $T_{B,n}$ is defined in

$$U_n := \{(y,t) \in \partial B \times (-\infty, 0] : \, -\sigma(y,n) \le t \le 0\},$$

where

$$\begin{aligned} \sigma(y,n) &:= \sup\{t : \, T_{B,n}(y,-t) \in \overline{B}\} = \sup\{t : \, T_{S^2,n}(y,-t) \in F_n(\overline{B})\}, \\ \sigma(\partial B, n) &:= \inf\{\sigma(y,n) : \, y \in \partial B\}. \end{aligned}$$

Differentiating (4.2) we find in $U_n \cap \partial B \times (-\varepsilon_0, 0]$

$$DF_n|T_{B,n} \circ DT_{B,n} = DT_{S^2,n}.$$

Hence, $T_{B,n}$ is a local diffeomorphism and it is enough to show, after possibly shrinking $\varepsilon_0 > 0$, that

$$\begin{aligned} \partial B \times (-\varepsilon_0, 0] \subset U_n & \text{ for all } n \in \mathbb{N}, \\ T_{B,n}|_{\partial B \times (-\varepsilon_0, 0]} & \text{ is injective.} \end{aligned}$$

To obtain a contradiction assume that

$$\sigma(\partial B, n) \to 0 \text{ as } n \to \infty.$$

It is standard to see that $\sigma(\partial B, n)$ is attained at some $y_{0,n} \in \partial B$ and that the geodesic

$$[0, \sigma(\partial B, n)] \ni t \mapsto T_{B,n}(y, -t)$$

is perpendicular to ∂B at $T_{B,n}(y, -\sigma(\partial B, n))$.
We fix $n_1 > 0$ such that $\sigma(\partial B, n_1) < \varepsilon_0/2$ and $\sigma(\partial B, n_1)$ is attained at $y_0 \in \partial B$. Due to the minimality of $\sigma(y_0, n_1)$ the parallel curve

$$y \mapsto T_{B,n_1}(y, \sigma(y_0, n_1))$$

lies inside $\overline{B}$ with positive curvature and touches ∂B at y_0 from the inside. This leads to the desired contradiction due to the positive curvature of ∂B and the maximum principle.
To show the injectivity we argue by contradiction and assume that there is a subsequence of (γ_n), still denoted by (γ_n), and a sequence $(y_{1,n}, y_{2,n}, t_{1,n}, t_{2,n})$ in $(\partial B)^2 \times (0, \frac{1}{n})^2$ such that $y_{1,n} \neq y_{2,n}$ and

$$T_{B,n}(y_{1,n}, -t_{1,n}) = T_{B,n}(y_{2,n}, -t_{2,n}).$$

Going to a subsequence we may assume

$$y_{1,n} \to y_1 \text{ and } y_{2,n} \to y_2 \text{ as } n \to \infty.$$

For $i \in \{1, 2\}$ we have

$$F_n \circ T_{B,n}(y_{i,n}, -t_{i,n}) = T_{S^2,n}(y_{i,n}, -t_{i,n}).$$

Consequently, $y_1 \neq y_2$, because $(T_{S^2,n})$ converges in $C^1(\partial B \times (-\varepsilon_0, \varepsilon_0), S^2)$ to $T_{S^2,0}$, which is an immersion. By the same argument, we deduce that for any $0 < r$ and any $0 < \varepsilon \leq \varepsilon_0$ there is $\delta = \delta(r, \varepsilon) > 0$ and $n_0 = n_0(r, \varepsilon)$ such that for all $n \geq n_0$

$$B_\delta(y_{1,n}) \subset T_{S^2,n}(B_{r,\partial B}(y_{1,n}) \times (-\varepsilon, \varepsilon)).$$

Since $0 < t_{1,n}, t_{2,n} < \frac{1}{n}$ and $y_1 \neq y_2$, taking $0 < r < \mathrm{dist}_{\partial B}(y_1, y_2)/2$ and $0 < \varepsilon \leq \varepsilon_0$ we infer that $y_2 \in B_\delta(y_1)$ for n large enough. Consequently, as $\varepsilon > 0$ may be chosen arbitrarily small,

$$\sigma(\partial B, n) \to 0 \text{ as } n \to \infty.$$

This yields a contradiction as above and finishes the proof. □

Lemma 4.4. *The set of regular, oriented Alexandrov embedded curves is open in $H^{2,2}(S^1, S^2)$.*

Proof. Let $\gamma_0 \in H^{2,2}(S^1, S^2)$ be oriented Alexandrov embedded. Then there is an oriented immersion $F_0 : \overline{B} \to S^2$ such that $F_0|_{\partial B} = \gamma$. We may extend F_0 to an open neighborhood U of $\overline{B}$ such that F_0 remains an immersion and $F_0(U)$ is open. If γ is close enough to γ_0, then γ lies in $F_0(U)$ and we may write γ as a graph over γ_0 with respect to its normal. Since $\gamma(\partial B) \subset F_0(U)$ we may proceed as in the proof of Lemma 4.2 to deduce that γ is oriented Alexandrov embedded as well. □

5. The apriori estimate

We fix a continuous family of metrics $\{g_t : t \in [0,1]\}$ on S^2 and a continuous family of positive continuous function $\{k_t : t \in [0,1]\}$ on S^2, such that the Gauß curvature K_{g_t} is nonnegative and

$$k_{inf} := \inf\{k_t(x) : (x,t) \in S^2 \times [0,1]\} > 0.$$

We let X_t be the vector field on $H^{2,2}(S^1, S^2)$ defined by

$$X_t := X_{k_t, g_t}.$$

We denote by $M_A \subset H^{2,2}(S^1, S^2)$ the set

$$M_A := \{\gamma \in H^{2,2}_{reg}(S^1, S^2) : \gamma \text{ is oriented Alexandrov embedded.}\}.$$

We shall show that the set

$$X^{-1}(0) := \{(\gamma, t) \in M_A \times [0,1] : X_t(\gamma) = 0\}$$

is compact in $M_A \times [0,1]$. Fix $(\gamma, t) \in X^{-1}(0)$. Then there is an oriented immersion $F : \overline{B} \to S^2$ with $F|_{\partial B} = \gamma$. We denote by $F^* g_t$ the induced metric on B. Using $F|_{\partial B} = \gamma$, $K_{g_t} \geq 0$, and the fact that F is a local isometry from $(B, F^* g_t)$ to (S^2, g_t), the Gauß-Bonnet formula gives

$$\begin{aligned} 2\pi &= \int_{\partial B} k_{F*g_t} \, dS_{F*g_t} + \int_B K_{F*g_t} \, dA_{F*g_t} \\ &\geq \int_\gamma k_t \, dS_{g_t} \geq k_{inf} L(\gamma, g_t), \end{aligned}$$

where $L(\gamma, g_t)$ denotes the length of γ in the metric g_t.
To obtain a contradiction assume that there is (γ_n, t_n) in $X^{-1}(0)$ such that $L(\gamma_n, g_{t_n}) \to 0$ as $n \to \infty$. We denote by F_n the corresponding oriented immersions. Since the sets $\{g_t : t \in [0,1]\}$ and $\{k_t : t \in [0,1]\}$ are compact, all involved metric are uniform equivalent to the standard metric g_{can} and there is $C_{k,g_{can}} > 0$, such that

$$|k_{g_{can}}(\gamma_n, t)| \leq C_{k,g_{can}} \text{ for all } (n,t) \in \mathbb{N} \times S^1.$$

For $n \in \mathbb{N}$ we let

$$\begin{aligned} L_n &:= L(\gamma_n, g_{can}) = \int_{\partial B} dS_{F_n^* g_{can}}, \\ \tilde{L}_n &:= \int_{\gamma_n} k_{g_{can}} dS_{g_{can}} = \int_{\partial B} k_{F_n^* g_{can}} dS_{F_n^* g_{can}}, \\ A_n &:= \int_B dA_{F_n^* g_{can}} = \int_B K_{F_n^* g_{can}} dA_{F_n^* g_{can}}. \end{aligned}$$

Due to the uniform equivalence of the involved metrics and the uniform bound on the geodesic curvature L_n and $\tilde{L}_n$ tend to 0 as $n \to \infty$. From the Gauß-Bonnet formula applied to $(B, F_n^* g_{can})$ we obtain

$$A_n = 2\pi - \tilde{L}_n.$$

Applying the isoperimetric inequality [8] to $(B, F_n^* g_{can})$ we find

$$\begin{aligned} L_n^2 &\geq 4\pi A_n - (A_n)^2 \\ &= 4\pi\big(2\pi - \tilde{L}_n\big) - \big(2\pi - \tilde{L}_n\big)^2 \\ &= 4\pi^2 - (\tilde{L}_n)^2, \end{aligned}$$

which is impossible for large n.
Consequently, the length $L(\gamma, g_t)$ for $(\gamma, t) \in X^{-1}(0)$ satisfies

$$c \leq L(\gamma, g_t) \leq \big(\inf\{k_t(x)\}\big)^{-1} 2\pi \tag{5.1}$$

for some positive constant $c = c(\{k_t\}, \{g_t\})$.
Fix a sequence $(\gamma_n, t_n)_{n\in\mathbb{N}}$ in $X^{-1}(0)$. Since γ_n is a zero of X_{t_n}, the curve γ_n is parameterized proportional to its arc-length. From the bound in (5.1) we see that (γ_n) is uniformly bounded in $C^1(S^1, S^2)$. Using the equation (1.2) and standard regularity theory we find that (γ_n) is bounded in $C^3(S^1, S^2)$, such that we may extract a subsequence, still denoted by $(\gamma_n, t_n)_{n\in\mathbb{N}}$, which converges in $C^2(S^1, S^2) \times [0, 1]$ to (γ_0, t_0). Due to the convergence in $C^2(S^1, S^2)$ we have $X_{t_0}(\gamma_0) = 0$, and the lower bound in (5.1) implies that γ_0 is an immersion. From the stability of oriented Alexandrov embeddings in Lemma 4.2 we deduce that γ_0 is oriented Alexandrov embedded and hence $(\gamma_0, t_0) \in X^{-1}(0)$. This shows that $X^{-1}(0)$ is compact.
From the homotopy invariance we now get

$$\chi_{S^1}(X_{k_0, g_{can}}, M_A) = \chi_{S^1}(X_{k_1, g_1}, M_A). \tag{5.2}$$

6. Existence results

We give the proof of our main existence result.

Proof of Theorem 1.2. From the uniformization theorem there are a function $\varphi \in C^\infty(S^2, \mathbb{R})$ and an isometry F of (S^2, g) to $(S^2, e^\varphi g_{can})$, where g_{can} denotes the standard round metric on S^2. Since the problem of prescribing geodesic curvature is invariant under isometries we may assume without loss of generality that

$$g = e^\varphi g_{can}.$$

We consider the family of metrics $\{g_t : t \in [0, 1]\}$ and the family of positive continuous function $\{k_t : t \in [0, 1]\}$ defined by

$$\begin{aligned} g_t &:= e^{t\varphi} g_{can}, \\ k_t &:= (1 - t)(\inf k) + tk. \end{aligned}$$

Then $k_t \geq \inf k > 0$ for all $t \in [0, 1]$ and the Gauß curvature K_{g_t} of the metric g_t satisfies

$$\begin{aligned} K_{g_t} &= e^{-t\varphi}\big(-t\Delta_{g_{can}}(\varphi) + 2\big) \\ &= e^{-t\varphi}\big(-t(2 - K_g e^\varphi) + 2\big) \geq 0, \end{aligned}$$

because K_g is nonnegative.
From Section 5 the homotopy

$$[0,1] \ni t \mapsto X_{k_t,g_t}$$

is (M_A, g_t, S^1)-admissible and by (3.6) and (5.2)

$$-2 = \chi_{S^1}(X_{k_0,g_{can}}, M_A) = \chi_{S^1}(X_{k,g}, M_A).$$

This gives the claim. □

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References

[1] V. I. Arnold. The first steps of symplectic topology. *Uspekhi Mat. Nauk*, 41(6(252)):3–18, 229, 1986.
[2] Vladimir I. Arnold. *Arnold's problems*. Springer-Verlag, Berlin, 2004. Translated and revised edition of the 2000 Russian original, With a preface by V. Philippov, A. Yakivchik and M. Peters.
[3] Gonzalo Contreras, Leonardo Macarini, and Gabriel P. Paternain. Periodic orbits for exact magnetic flows on surfaces. *Int. Math. Res. Not.*, (8):361–387, 2004.
[4] V. L. Ginzburg. New generalizations of Poincaré's geometric theorem. *Funktsional. Anal. i Prilozhen.*, 21(2):16–22, 96, 1987.
[5] Viktor L. Ginzburg. On closed trajectories of a charge in a magnetic field. An application of symplectic geometry. In *Contact and symplectic geometry (Cambridge, 1994)*, volume 8 of *Publ. Newton Inst.*, pages 131–148. Cambridge Univ. Press, Cambridge, 1996.
[6] Viktor L. Ginzburg. On the existence and non-existence of closed trajectories for some Hamiltonian flows. *Math. Z.*, 223(3):397–409, 1996.
[7] S. P. Novikov and I. A. Taimanov. Periodic extremals of multivalued or not everywhere positive functionals. *Dokl. Akad. Nauk SSSR*, 274(1):26–28, 1984.
[8] Robert Osserman. The isoperimetric inequality. *Bull. Amer. Math. Soc.*, 84(6):1182–1238, 1978.
[9] Frank Quinn and Arthur Sard. Hausdorff conullity of critical images of Fredholm maps. *Amer. J. Math.*, 94:1101–1110, 1972.
[10] Felix Schlenk. Applications of Hofer's geometry to Hamiltonian dynamics. *Comment. Math. Helv.*, 81(1):105–121, 2006.
[11] Matthias Schneider. Closed magnetic geodesics on S^2. Preprint, `arXiv:0808.4038 [math.DG]`, 2008.
[12] S. Smale. An infinite dimensional version of Sard's theorem. *Amer. J. Math.*, 87:861–866, 1965.
[13] I. A. Taimanov. Non-self-intersecting closed extremals of multivalued or not-everywhere-positive functionals. *Izv. Akad. Nauk SSSR Ser. Mat.*, 55(2):367–383, 1991.
[14] I. A. Taimanov. Closed extremals on two-dimensional manifolds. *Uspekhi Mat. Nauk*, 47(2(284)):143–185, 223, 1992.
[15] A. J. Tromba. A general approach to Morse theory. *J. Differential Geometry*, 12(1):47–85, 1977.
[16] A. J. Tromba. The Euler characteristic of vector fields on Banach manifolds and a globalization of Leray-Schauder degree. *Adv. in Math.*, 28(2):148–173, 1978.

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